DUSO Mathematics League 2015 - 2016

CONTEST #2.

SOLUTIONS

2 - 1. $\boxed{\mathbf{24}}$ Substitute $x = 1$ to obtain $y \leq 12$, so there are 12 ordered pairs with $x = 1$. Similarly, there are 8 ordered pairs with $x = 2$ and 4 with $x = 3$, and no others with positive coordinates. The answer is $12 + 8 + 4 = \mathbf{24}$.

2 - 2. $\boxed{\mathbf{43}}$ The only units digits whose sixth powers end in 9 are 3 and 7. Note that $40^6 = 4,096,000,000$ and $50^6 = 15,625,000,000$, so $40 < N < 50$, and since N^6 is closer to 40^6 than 50^6, N must be **43**. To check the answer, consider that 43^2 ends in 49. $49^2 = 43^4$ ends in 01. Therefore, $49^3 = 43^6$ ends in 49, as needed.

---

2 - 3. $\boxed{(\mathbf{10}, \mathbf{5})}$ Suppose the four vertices of $SQUA$ are $S(7, 1)$, $Q(3, 4)$, $U(6, 8)$, and $A(x, y)$. Then, because U is the image of Q after a translation along the vector $< 3, 4 >$, A must be the image of S after a translation along the same vector, so A has coordinates $(7 + 3, 1 + 4) = (\mathbf{10}, \mathbf{5})$.

2 - 4. $\boxed{\sqrt{\mathbf{34}}}$ The base has side length $24 \div 4 = 6$ cm. The volume is $60 = \frac{1}{3}(6^2)(h) \rightarrow h = 5$ cm. The height H of the isosceles triangular side is the hypotenuse of a right triangle whose legs are $h = 5$ and $6 \div 2 = 3$, so $H = \sqrt{3^2 + 5^2} = \sqrt{\mathbf{34}}$.

---

2 - 5. $\boxed{\mathbf{28}}$ Let the number of students in the class be N; then $N(N - 1)/2 - (N - 4)(N - 5)/2 = 102 \rightarrow N^2 - N - (N^2 - 9N + 20) = 204$, which solves to obtain $8N - 20 = 204 \rightarrow N = \mathbf{28}$.

2 - 6. $\boxed{\mathbf{-16}}$ Recall that $(A + B)^4 = A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4$. Therefore, $(A - B)^4 + (A + B)^4 = (A^4 - 4A^3B + 6A^2B^2 - 4AB^3 + B^4) + (A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4)$, or $2A^4 + 12A^2B^2 + 2B^4$. Substituting $A = \sqrt{3}$ and $B = i$, the desired value is $2 \cdot 3^2 + 12 \cdot 3 \cdot -1 + 2 \cdot 1 = 18 - 36 + 2$, or $\mathbf{-16}$.

Author: George Reuter - coachreu@gmail.com – Reviewer: Michael Curry - currymath@gmail.com

T-1. Sammy and Tammy go out for a one-hour jog. Sammy alternates between running for 5 minutes at 6 miles per hour and then walking for 1 minute at 2 miles per hour. Tammy runs at a constant pace of M miles per hour. Sammy and Tammy finish their jog at the same time. Compute M.

T-1Sol. $\boxed{\mathbf{\frac{16}{3}}}$ Convert to one-hour, so that Sammy runs for $\frac{5}{6} \cdot 60 = 50$ minutes and walks for 10 minutes. Running for 50 minutes at 6 miles per hour converts to a distance of $\frac{5}{6} \cdot 6 = 5$ miles. Similarly, Sammy walks for $\frac{1}{6} \cdot 2 = \frac{1}{3}$ mile in the hour. The value of M is $5 + \dfrac{1}{3} = \mathbf{\dfrac{16}{3}}$.

T-2. Compute the value of the greatest integer N such that 7^N divides 2015!.

T-2Sol. $\boxed{\mathbf{333}}$ There is one factor of 7 in 2015! for each of 7, 14, 21, ..., $7 \cdot 287 = 2009$. There is an additional factor of 7 in 2015! for each of 49, 98, ..., $49 \cdot 41 = 2009$. There is an additional factor of 7 in 2015! for each of 343, 686, ..., $343 \cdot 5 = 1715$. The value of N is $287 + 41 + 5 = \mathbf{333}$.

T-3. Let $f(x) = x^3 - 6x^2 + 8x - 5$. If $f(x)$ can also be expressed $f(x) = (x-2)^3 + b(x-2)^2 + c(x-2) + d$, compute the ordered triple (b, c, d).

T-3Sol. $\boxed{(\mathbf{0}, \mathbf{-4}, \mathbf{-5})}$ Let $u = x - 2$. Then $f(u) = u^3 + bu^2 + cu + d = (u+2)^3 - 6(u+2)^2 + 8(u+2) - 5$, or $u^3 + 6u^2 + 12u + 8 - 6u^2 - 24u - 24 + 8u + 16 - 5$, so $b = 0$, $c = -4$, and $d = -5$. The ordered triple is $(\mathbf{0}, \mathbf{-4}, \mathbf{-5})$.

Author: George Reuter - coachreu@gmail.com – Reviewer: Michael Curry - currymath@gmail.com